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Heat transfer in a tube with pulsating flow and constant heat flux

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Abstract—The problem of pulsatile flow in a tube with constant heat flux at the wall is considered analytically to determine how pulsation affects the rate of heat transfer and how the phenomenon depends on the Prandtl number and on pulsation frequency. The results indicate that in a range of moderate values of the frequency there is a positive peak in the effect of pulsation whereby the bulk temperature of the fluid and the Nusselt number are increased, but the effect is reversed when the frequency is outside this range. The peaks are higher at lower Prandtl numbers.

INTRODUCTION

The problem of pulsatile flow in a tube has received considerable attention in recent years. The classical work of Womersley [1], Uchida [2] and Atabek and Chang [3] and many others that followed have led to exact solutions and considerable information about the oscillatory flow field. Studies of the corresponding heat transfer problem, however, have been far less numerous and existing results do not provide complete information about the temperature field and the rate of heat transfer in pulsatile flow. The problem is important in biological applications in relation to blood flow and in industrial applications in relation to heat exchange efficiency.

In general the pulsatile flow field is assumed to consist of a steady Poiseuille flow part and a purely oscillatory part, and it is believed that the rate of heat transfer is changed because pulsation alters the thickness of the thermal boundary layer and hence the thermal resistance. This view was first supported by Richardson [4] who showed that the velocity profile for pulsating flow is steeper near the wall than it is in steady Poiseuille flow. It then follows from consideration of Reynolds analogy that the temperature profile will be affected in a similar way and the rate of heat transfer should increase.

Later, Siegel and Perlmutter [5] demonstrated the explicit dependence of overall heat transfer on pulsation frequency. They found that when constant temperatures was prescribed for the wall boundary condition, the local Nusselt number showed periodic behaviour which could enhance heat transfer. More recently, Creff and Andre [6] studied developing pulsatile flow in a duct using finite difference methods and showed the importance of the entry region for the unsteady thermal phenomena and their longitudinal evolution. Cho and Hyun [7] obtained a numerical solution of the boundary layer equations coupled with a corresponding form of the energy equation for pulsatile flow with heat transfer in a pipe. They found an increase in Nusselt number with an increase in the amplitude of pulsation as well as frequency, but in the latter the increase in Nusselt number had characteristics of a local maximum. Later, Kim *et al.* [8] studied the corresponding heat transfer problem in a channel.

The aim of the present paper is to demonstrate the mechanics of heat transfer in pulsatile flow in a pipe by means of an analytical solution in order to show clearly how pulsation affects the rate of heat transfer and how the phenomenon depends on the Prandtl number and on the nondimensional frequency parameter. For this reason we choose the problem of pulsatile flow with constant heat flux at the tube wall. This boundary condition allows us to solve the coupled partial differential equations involved and obtain a measure of the bulk temperature within the tube. The condition of constant heat flux may be particularly relevant in biological applications where heat is generated at a constant rate as a result of metabolic activity. It may also be important in heat exchangers in which the exchange is controlled to occur at a constant rate.

GOVERNING EQUATIONS

The temperature T^* within the tube is assumed to be governed by the simplified energy equation

$$\frac{\partial T^*}{\partial t^*}+u^*\frac{\partial T^*}{\partial x^*}=\frac{k}{\rho c_p}\left(\frac{\partial^2 T^*}{\partial r^{*2}}+\frac{1}{r^*}\frac{\partial T^*}{\partial r^*}\right) \quad (1)$$

where c_p, k, ρ are specific heat, conductivity and den-

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NOMENCLATURE

a	tube radius
c_p	heat capacity at constant pressure
i	$=\sqrt{-1}$
J_0, Y_0	Bessell functions of the first and second kind of order 0
k	thermal conductivity of the fluid
Nu	Nusselt number
p	pressure
Pr	Prandtl number
q_w	heat transfer at the tube wall
r	radial coordinate measured from tube axis
Re	Reynolds number
t	time
T	temperature
u	velocity component in axial direction
x	axial coordinate measured from tube entrance.

Greek symbols

α	$=\sqrt{-i\omega}$
β	nondimensional pulsation parameter
θ	temperature difference
θ_{1b}	oscillatory bulk temperature
λ	$=\alpha/\sqrt{Pr}$
μ, ρ	viscosity and density of the fluid
ω	frequency
$*$	denotes a dimensional quantity.

sity of the fluid, respectively, x^*, r^* are axial and radial coordinates within the tube, t^* is time, and a star is being used to indicate the dimensional form of these variables. As the form of this equation indicates, the flow is assumed to be in the axial direction only so that velocity components in the other two directions are zero. Also, viscous dissipation effects are assumed to be negligibly small compared with the convective rate of heat transfer so that the dissipation function term in the full energy equation is neglected.

The solution sought is for a region of the tube which we shall refer to as the "thermal region", where the heat flux at the tube wall is constant and fluid is entering the region at a uniform temperature T_0^*. Introducing the nondimensional temperature difference

$$\theta = \frac{T^*-T_0^*}{q_w a/k} \tag{2}$$

where q_w is the constant heat flux at the tube wall and a is the radius of the tube, other variables are nondimensionalized as follows

$$\begin{aligned} x &= x^*k/\rho c_p \hat{u} a^2 = x^*/aPrRe \\ t &= t^*k/\rho c_p a^2 \\ r &= r^*/a \\ u &= u^*/\hat{u}_0. \end{aligned} \tag{3}$$

Here $\hat{u}_0$ is a constant reference velocity to be specified later and

$$\begin{aligned} Pr &= \frac{\mu c_p}{k} \\ Re &= \frac{\rho \hat{u}_0 a}{\mu}. \end{aligned} \tag{4}$$

With these the governing equation takes the nondimensional form

$$\frac{\partial \theta}{\partial t}+u\frac{\partial \theta}{\partial x} = \frac{\partial^2 \theta}{\partial r^2}+\frac{1}{r}\frac{\partial \theta}{\partial r} \tag{5}$$

with boundary conditions for $\theta(x, r, t)$

$$\begin{aligned} \frac{\partial \theta}{\partial r}(x,1,t) &= 1 \\ \theta(0,r,t) &= 0 \\ \frac{\partial \theta}{\partial r}(x,0,t) &= 0. \end{aligned} \tag{6}$$

OSCILLATORY FLOW FIELD

The flow field within the thermal region of the tube is now assumed to consist of a steady part represented by fully developed Poiseuille flow [9] plus an oscillatory part represented by the classical solution for pulsatile flow in a rigid tube [1, 2]. Thus in dimensional form we write

$$u^*(r,t) = \frac{A_0}{4\mu}(r^{*2}-a^2)+\frac{iA_1}{\rho\omega^*} \times\left(1-\frac{J_0(\sqrt{-i\rho\omega^*a^2/\mu}\,r^*/a)}{J_0(\sqrt{-i\rho\omega^*a^2/\mu})}\right)e^{i\omega t} \tag{7}$$

where A_0 is the pressure gradient driving the steady part of the flow and A_1 is the amplitude of the oscillatory pressure gradient driving the oscillatory part of the flow, that is

$$\frac{\partial p^*}{\partial x^*} = A_0+A_1 e^{i\omega^* t^*} \tag{8}$$

where ω^* is dimensional frequency, $i=\sqrt{-1}$, and J_0 is a Bessel function of the first kind of order 0. It should be noted that both parts of the velocity field are independent of x, that is both the steady and the pulsatile parts of the flow are assumed to be fully developed.

Introducing a nondimensional frequency parameter

$$\omega = \frac{\rho c_p a^2}{k}\omega^* = PrRe_\omega \quad (9)$$

where $R_\omega = \rho a(a\omega^*)/\mu$ is a Reynolds number based on the velocity $a\omega^*$, the expression (equation (7)), for the velocity can be put in the nondimensional form

$$u(r,t) = u_0(r)+\beta u_1(r,t) \quad (10)$$

where

$$u_0(r) = (1-r^2) \quad (11)$$

$$u_1(r,t) = \frac{-i}{\omega}\left(1-\frac{J_0(\sqrt{-i\omega/Pr}\,r)}{J_0(\sqrt{-i\omega/Pr})}\right)e^{i\omega t} \quad (12)$$

$$\beta = 4\left(\frac{A_1}{A_0}\right)Pr. \quad (13)$$

Variables have been nondimensionalized as before, with the normalizing velocity $\hat{u}_0$ now being taken as the maximum velocity in Poiseuille flow, namely

$$u_0 = \frac{u_0^*}{\hat{u}_0},\quad u_1 = \frac{u_1^*}{\hat{u}_0},\quad \hat{u}_0 = \frac{-A_0 a^2}{4\mu}. \quad (14)$$

A solution for θ is now sought in the same form as that of the velocity (equation (10)), namely

$$\theta(x,r,t) = \theta_0(x,r)+\beta\theta_1(r,t) \quad (15)$$

where $\theta_0(x,r)$ represents the steady part of the temperature corresponding to that in steady flow, while $\theta_1(r,t)$ represents the change in temperature due to oscillatory flow. The latter is independent of x since, as discussed earlier, it is based on fully developed flow. This form for the temperature field must clearly rest on the assumption that the value of the nondimensional parameter β is small, or in any case does not exceed 1.0, but again in the fully developed region of the flow this condition can be somewhat relaxed.

Substituting for the velocity and temperature in the governing equation (equation (5)) yields the following two boundary value problems for $\theta_0(x,r)$ and $\theta_1(r,t)$

$$u_0\frac{\partial\theta_0}{\partial x} = \frac{\partial^2\theta_0}{\partial r^2}+\frac{1}{r}\frac{\partial\theta_0}{\partial r}$$

$$\frac{\partial\theta_0}{\partial r}(x,1) = 1$$

$$\theta_0(0,r) = 0$$

$$\frac{\partial\theta_0}{\partial r}(x,0) = 0 \quad (16)$$

$$\frac{\partial\theta_1}{\partial t}+u_1\frac{\partial\theta_0}{\partial x} = \frac{\partial^2\theta_1}{\partial r^2}+\frac{1}{r}\frac{\partial\theta_1}{\partial r}$$

$$\frac{\partial\theta_1}{\partial r}(1,t) = 0$$

$$\frac{\partial\theta_1}{\partial r}(0,t) = 0. \quad (17)$$

OSCILLATORY TEMPERATURE

A solution for the steady-state temperature θ_0 was obtained by Sellars *et al.* [10]. The form of the solution is rather complicated for calculation purposes but it is possible to deduce from it that θ_0 becomes a linear function of x downstream. More accurately, as x becomes large,

$$\frac{\partial\theta_0}{\partial x}\to 4.0. \quad (18)$$

In fact to a good degree of approximation the condition applies for $x > 1.0$.

This result makes it possible to seek a solution of equation (17) for $\theta_1(r,t)$ in the form

$$\theta_1(r,t) = 4h(r)\,e^{i\omega t} \quad (19)$$

which will be valid in the downstream region of the flow where equation (18) is valid. Substituting this into equation (17) leads to the following nonhomogeneous Bessel equation for $h(r)$

$$\frac{d^2h}{dr^2}+\frac{1}{r}\frac{dh}{dr}-i\omega h = g(r)$$

$$h'(1) = 0, h'(0) = 0 \quad (20)$$

where

$$g(r) = \frac{-i}{\omega}\left(1-\frac{J_0(\sqrt{-i\omega/Pr}\,r)}{J_0(\sqrt{-i\omega/Pr})}\right). \quad (21)$$

We obtain a solution of equation (20) in terms of Green's functions which has the general form [11]

$$h(r) = \frac{-c_1}{c}J_0(\lambda r)\int_0^1 J_0(\lambda\tau)g(\tau,\omega)\,d\tau + \int_0^1 G(r,\tau)g(\tau,\omega)\,d\tau \quad (22)$$

where

$$\lambda = \alpha/\sqrt{Pr}, \alpha = \sqrt{-i\omega} \quad (23)$$

$$c_1 = \mathscr{R}(b/a) \quad (24)$$

$$a = \left(\frac{\partial J_0(\alpha r)}{\partial r}\right)_{r=1}, b = \left(\frac{\partial Y_0(\alpha r)}{\partial r}\right)_{r=1} \quad (25)$$

where Y_0 is a Bessel function of the second kind of order 0, $\mathscr{R}$ denotes the real part of a complex quantity,

c is the Wronskian and G is the Green's function, defined by

$$c = \begin{vmatrix} J_0(\alpha r) & Y_0(\alpha r) \\ \alpha\frac{\partial J_0}{\partial r} & \alpha\frac{\partial Y_0}{\partial r} \end{vmatrix} \quad (26)$$

$$G(r,\tau) = \frac{1}{c}[Y_0(\alpha r)J_0(\alpha\tau)H(r-\tau) + J_0(\alpha r)Y_0(\alpha\tau)H(\tau-r)]. \quad (27)$$

After substitution for a, b, c, g and G, the solution takes the full form

$$h(r) = \frac{-\pi c_1}{2}\alpha J_0(\alpha r)\int_0^1 \frac{J_0(\alpha\tau)}{i\omega}\left[1-\frac{J_0(\lambda\tau)}{J_0(\lambda)}\right]\mathrm{d}\tau + \frac{\pi}{2}\alpha J_0(\alpha r)\int_r^1 \frac{Y_0(\alpha\tau)}{i\omega}\left[1-\frac{J_0(\lambda\tau)}{J_0(\lambda)}\right]\mathrm{d}\tau + \frac{\pi}{2}\alpha Y_0(\alpha r)\int_0^r \frac{J_0(\alpha\tau)}{i\omega}\left[1-\frac{J_0(\lambda\tau)}{J_0(\lambda)}\right]\mathrm{d}\tau. \quad (28)$$

Using this result, the oscillatory temperature $\theta_1(r, t)$, as defined in equation (19), is now fully determined.

RESULTS AND DISCUSSION

Because of the simplification (equation (18)) used to obtain the solution for the temperature, our results are restricted to the downstream region of the flow where the Nusselt numbers, steady and oscillatory, have become independent of x. Thus to determine the effect of pulsation on the rate of heat transfer in that region, we now define the local Nusselt number

$$Nu = \frac{q_w 2a/k}{\overline{\mathscr{R}T^*(1)} - T_b^*} \quad (29)$$

and the corresponding Nusselt number in steady flow

$$Nu_0 = \frac{q_w 2a/k}{T_0^*(1) - T_{0b}^*} \quad (30)$$

where subscript b refers to bulk properties, and an overbar indicates a time average over one cycle of oscillation, that is

$$\overline{\mathscr{R}T^*(1)} = \frac{1}{2\pi/\omega}\int_0^{2\pi/\omega} \mathscr{R}T^*(1,t)\,\mathrm{d}t \quad (31)$$

$$T_b = \frac{\int_0^1 \overline{\mathscr{R}u(r)\mathscr{R}T^*(r)}r\,\mathrm{d}r}{\int_0^1 \overline{\mathscr{R}u(r)}r\,\mathrm{d}r}. \quad (32)$$

In terms of the nondimensional temperature difference θ (equation (2)), the expressions for the Nusselt numbers (equations (29), (30)) become

$$Nu = \frac{2}{\overline{\mathscr{R}\theta(1)} - \theta_b} \quad (33)$$

$$Nu_0 = \frac{2}{\theta_0(1) - \theta_{0b}}. \quad (34)$$

Because of the periodicity of u_1, θ_1, and since u_0 represents the velocity profile in steady Poiseuille flow, it is found that

$$\int_0^1 \overline{\mathscr{R}u(r)}r\,\mathrm{d}r = \int_0^1 u_0(r)r\,\mathrm{d}r = \tfrac{1}{4} \quad (35)$$

and

$$\theta_b = \theta_{0b} + \beta^2\theta_{1b} \quad (36)$$

where

$$\theta_{0b} = \int_0^1 u_0\theta_0 r\,\mathrm{d}r \quad (37)$$

$$\theta_{1b} = 4\int_0^1 \overline{\mathscr{R}u_1\mathscr{R}\theta_1}\,r\,\mathrm{d}r = 8\int_0^1 (\mathscr{R}g(r)\mathscr{R}h(r) + \mathscr{I}g(r)\mathscr{I}h(r))r\,\mathrm{d}r \quad (38)$$

and $\mathscr{I}$ refers to the imaginary part of a complex quantity.

As a measure of the effect of pulsation we now consider the relative difference quantity

$$\Delta Nu = \frac{Nu - Nu_0}{Nu_0} = \frac{Nu}{Nu_0} - 1. \quad (39)$$

Substituting for the Nusselt numbers from equations (33) and (34), and for the bulk temperature from equation (36), this becomes finally

$$\Delta Nu = \frac{1}{\dfrac{2}{\beta^2 Nu_0\theta_{1b}} - 1}. \quad (40)$$

In this expression the parameter β is specified in accordance with its definition in equation (13). We have taken three values to illustrate the results: $\beta = 0.1, 0.5, 1.0$. The steady flow Nusselt number Nu_0 is taken from available results: we used $Nu_0 = 4.5$ from Sellars *et al.* [10]. The main variable in the calculation of ΔNu is thus the oscillatory bulk temperature θ_{1b}. This is consistent with the physics of the problem since heat transfer from the tube is constant because of the imposed boundary conditions, thus the effect of pulsation can occur only in terms of change in bulk temperature of fluid within the tube relative to wall temperature.

Values of the oscillatory bulk temperature θ_{1b} are calculated from equations (38) and (19), and the solution for $h(r)$ in equation (28). The calculation is not straightforward since the integrals in equation (28) must be evaluated analytically as functions of r before the integrals in equation (38) can be evaluated numeri-

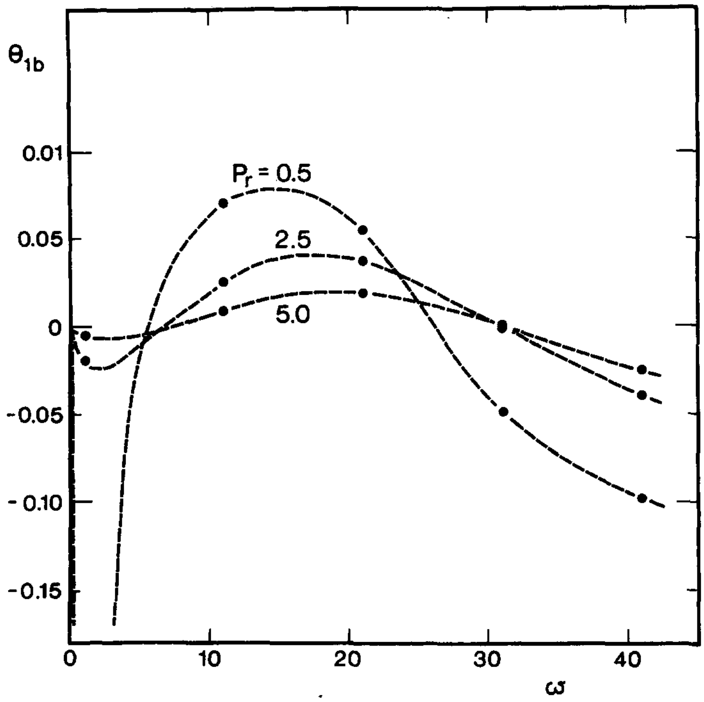

Fig. 1. Variation of bulk temperature θ_{1b} with nondimensional frequency ω and Prandtl number Pr.

cally. Bessel functions were replaced by appropriate Taylor series in order to achieve this.

Variation of θ_{1b} with frequency and with three different Prandtl numbers is shown in Fig. 1. It is observed that in each case the bulk temperature has a maximum at a certain value of the frequency, the maximum being higher at lower values of the Prandtl number. The corresponding change in Nusselt number is illustrated in Fig. 2, in terms of the variation of ΔNu with frequency. Again, a maximum is observed, which is higher at lower values of the Prandtl number. Effect of change in the pulsation parameter β is illustrated in Fig. 3. With $\beta = 0.1$ the effect of pulsation is close to zero, but with $\beta = 1.0$ pulsation can produce sig-

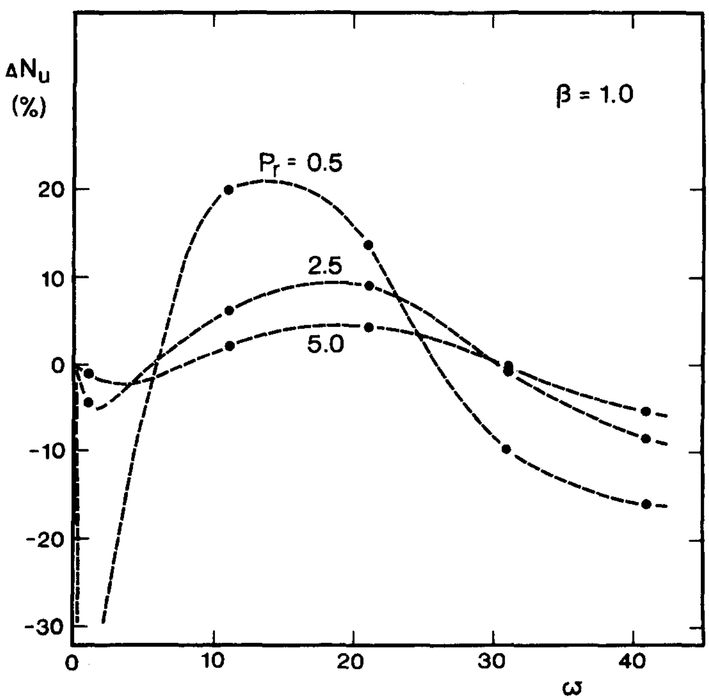

Fig. 2. Change in Nusselt number ΔNu with nondimensional frequency ω and Prandtl number Pr, with frequency parameter $\beta = 1.0$.

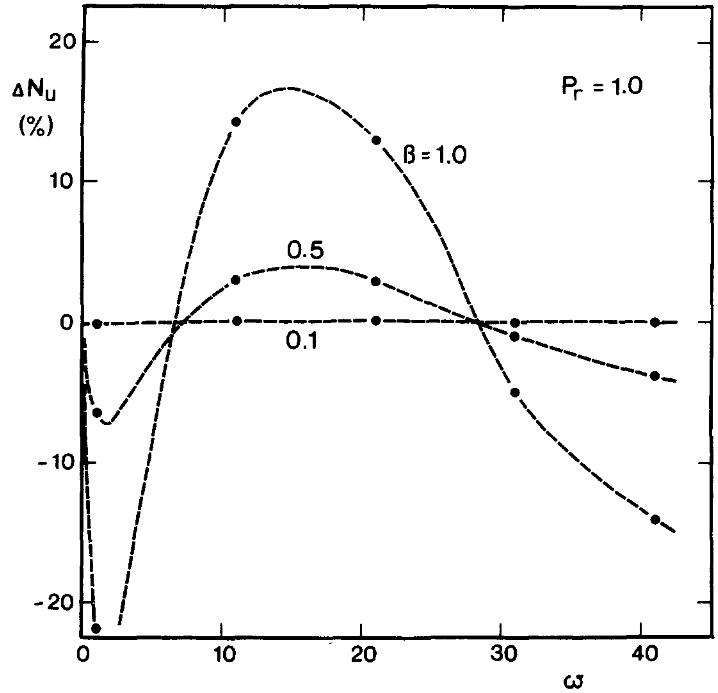

Fig. 3. Change in Nusselt number ΔNu with nondimensional frequency ω and frequency parameter β, with Prandtl number $Pr = 1.0$.

nificant change in the rate of heat transfer. Since this value of β may be at the limit of validity of the results, we use it here mainly to establish a trend rather than to produce accurate results at this value of β.

To conclude, our results indicate that the effect of pulsation on heat transfer in a tube with constant heat flux at the wall is to alter the bulk temperature of the fluid within the tube, thus allowing the constant heat flux to take place with a smaller or larger temperature difference than it does in steady flow. The results indicate that when $\omega \approx 15$ there is a positive peak in the effect of pulsation whereby the bulk temperature of the fluid and the Nusselt number are increased, but when the value of ω is (approximately) lower than 5 or higher than 25 this effect is reversed. The peaks are higher at lower Prandtl numbers. We believe that the reason for the characteristic peak in the curves is that the mechanism for change in the rate of heat transfer, as outlined by Richardson [4], is not effective at either very low or very high frequency. The reason for which the peaks are higher at lower Prandtl numbers P_r is that lower values of P_r are associated with lower momentum diffusivity and higher heat diffusivity. In general trend, these results are consistent with the conclusions of Barnett and Vachon [12] and those of Cho and Hyun [7].

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REFERENCES

1. Womersley, J. R., Oscillatory motion of a viscous liquid in a thin-walled elastic tube—I: The linear approximation for long waves. *Philosophical Magazine*, 1955, **46**, 199–221.
2. Uchida, S., The pulsating viscous flow superimposed on the steady laminar motion of incompressible fluid in a circular pipe. *ZAMP*, 1956, **7**, 403–422.
3. Atabek, H. B. and Chang, C. C., Oscillatory flow near the entry of a circular tube. *ZAMP*, **12**, 1961, 185–201.
4. Richardson, E. G., The transverse velocity gradient near the mouths of pipes in which an alternating or continuous flow of air is established. *Proceedings of the Royal Society*, 1929, **A42**, 1–15.
5. Siegel, R. and Perlmutter, M., Heat transfer for pulsating laminar duct flow. *Transactions ASME, Journal of Heat Transfer*, 1962, **84**, 111–123.
6. Creff, R. and Andre, P., Dynamic and convective results for a developing laminar unsteady flow. *International Journal of Numerical Methods in Fluids*, 1985, **5**, 745–760.
7. Cho, H. W. and Hyun, J. M., Numerical solutions of pulsating flow and heat transfer characteristics in a pipe. *International Journal of Heat Fluid Flow*, 1990, **11**, 321–330.
8. Kim, S. Y., Kang, B. H. and Hyun, J. M., Heat transfer in the thermally developing region of a pulsating channel flow. *International Journal of Heat and Mass Transfer*, 1993, **36**, 4257–4266.
9. Schlichting, H., *Boundary Layer Theory*, 7th edn. McGraw-Hill, New York, 1979, pp. 11–12.
10. Sellars, J. R., Tribus, M. and Klein, J. S., Heat transfer to laminar flow in a round tube or flat conduit—the Graez problem extended. *Transactions ASME*, 1956, **78**, 441–447.
11. Roach, G. F., *Green's Functions*. Cambridge University Press, 1982, Chapter 7, pp. 141–186.
12. Barnett, D. O. and Vachon, R. I., An analysis of convective heat transfer for pulsating flow in a tube. *Proceedings of the Fourth International Heat Transfer Conference*, FC 9.1, Paris, 1970, pp. 1–11.